Long Division of Polynomials

Topics

In this document we will review

- How to divide polynomials by other polynomials.

Why do we need this?

Here are two typical problems from MTH 132 that can/should be solved using long division of polynomials.

Example 1 (From Section 1.6)**.** Evaluate $\lim_{x\to 3} \dfrac{x^3-27}{x-3}$

Example 2 (From Section 3.5)**.** Determine if $g(x) = \dfrac{2x^2-3x}{4x+2}$ has any slant asymptotes. If it does, find them.

Important Definitions and Theorems

Theorem 3 (Division Algorithm)**.** If $f(x)$ and $d(x)$ are polynomials, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$, so that

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

and so that the degree of $r(x)$ is less than the degree of $d(x)$. In the special case where $r(x) = 0$, we say that $d(x)$ divides evenly into $f(x)$.

Definition 4. In the above theorem $q(x)$ is called the **quotient**, $r(x)$ is called the **remainder**, and $d(x)$ is called the **divisor**.

Instructional Videos

Click on the following links to access helpful instructional videos:

- Super Review: Long division of integers
  `www.math.msu.edu/classes/mth_132/review_video/div1.aspx`
- Introduction into polynomial long division
  `www.math.msu.edu/classes/mth_132/review_video/div2.aspx`
- More difficult polynomial division
  `www.math.msu.edu/classes/mth_132/review_video/div3.aspx`

Now that you have been exposed to all the ideas and seen a couple solutions worked out you should try a few problems. Please see the quiz which has some questions for you to try and the answers posted at the end. The important thing is the work that leads to the answers. That's where you come in!

Quiz

1. Express $\frac{4101}{12}$ as a mixed number.

$$
\begin{array}{r}
341 \\
12 \,\overline{)\,4101} \\
\underline{36} \\
50 \\
\underline{48} \\
21 \\
\underline{12} \\
9
\end{array}
$$

341 ← quotient

9 ← remainder

$$4101 = 341 \cdot 12 + 9$$

$$\frac{4101}{12} = 341 + \frac{9}{12}$$

$$= 341\frac{9}{12}$$

2. Find the quotient $q(x)$ and remainder $r(x)$ for the following rational expression: $\frac{4x^3 + x^2 + x + 1}{x - 2}$

$$
\begin{array}{r}
4x^2 + 9x + 19 \\
x-2 \,\overline{)\,4x^3 + x^2 + x + 1} \\
\underline{4x^3 - 8x^2 } \\
9x^2 + x \\
\underline{9x^2 - 18x } \\
19x + 1 \\
\underline{19x - 38} \\
39
\end{array}
$$

$4x^2 + 9x + 19$ ← $q(x)$

39 ← $r(x)$

$$4x^3 + x^2 + x + 1 = (4x^2 + 9x + 19)(x-2) + 39$$

$$\Rightarrow \frac{4x^3 + x^2 + x + 1}{x-2} = \boxed{4x^2 + 9x + 19} + \frac{\boxed{39}}{x-2}$$

3. Find the quotient $q(x)$ and remainder $r(x)$ for the following rational expression: $\dfrac{2x^2}{1-x}$

$$\begin{array}{r|l} & -2x \quad -2 \\ \hline -x+1 & 2x^2+0+0 \\ & 2x^2-2x \\ \hline & \quad 2x+0 \\ & \quad 2x-2 \\ \hline & \qquad 2 \end{array}$$

$$2x^2=(-2x-2)(-x+1)+2$$

$$\Rightarrow \frac{2x^2}{-x+1}=\boxed{-2x-2}+\frac{\boxed{2}}{-x+1}.$$

We would like to rewrite $1-x$ as $-x+1$. Rank the order from high to low.

Remark: there are no linear and constant terms for x^2. We need to fill in 0 at the corresponding positions.

4. Find the quotient $q(x)$ and remainder $r(x)$ for the following rational expression: $\dfrac{x^4-2x^3+1}{x^2}$

Remark: the denominator is a single term x^2, we do not need to apply long division. It is enough to split the numerator directly.

$$\frac{x^4-2x^3+1}{x^2}=\frac{x^4}{x^2}-\frac{2x^3}{x^2}+\frac{1}{x^2}$$

$$=\boxed{x^2-2x}+\frac{\boxed{1}}{x^2}$$

5. Find the quotient $q(x)$ and remainder $r(x)$ for the following rational expression: $\dfrac{x^3-1}{x^2-x-2}$

$$\begin{array}{r}
x+1 \\
x^2-x-2 \;\big)\; \overline{x^3+0+0-1} \\
\underline{x^3-x^2-2x} \\
x^2+2x-1 \\
\underline{x^2-x-2} \\
3x+1
\end{array}$$

$x^3-1=(x+1)(x^2-x-2)+(3x+1)$

$\Rightarrow \dfrac{x^3-1}{x^2-x-2}=\boxed{x+1}+\dfrac{\boxed{3x+1}}{x^2-x-2}$.

6. Find the quotient $q(x)$ and remainder $r(x)$ for the following rational expression: $\dfrac{(2x+3)(x-1)}{x^2-1}$

Remark: expand the numerator first.

$(2x+3)\cdot(x-1)=2x^2+x-3$

$$\begin{array}{r}
2 \\
x^2+0-1 \;\big)\; \overline{2x^2+x-3} \\
\underline{2x^2+0-2} \\
-1
\end{array}$$

$\Rightarrow 2x^2+x-3=2\cdot(x^2-1)-1$

$\dfrac{2x^2+x-3}{x^2-1}=\boxed{2}+\dfrac{\boxed{-1}}{x^2-1}$.

Remark2: We can factorize the denominator first.

$x^2-1=(x+1)\cdot(x-1)$.

$\dfrac{(2x+3)(x-1)}{x^2-1}=\dfrac{(2x+3)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}=\dfrac{2x+3}{x+1}=\dfrac{(2x+2)+1}{x+1}=2-\dfrac{1}{x+1}$.

7. Simplify $\dfrac{x^3-27}{x-3}$

$$\begin{array}{r|l} & x^2+3x+9 \\ \hline x-3 & x^3+0+0-27 \\ & x^3-3x^2 \\ \hline & 3x^2+0 \\ & 3x^2-9x \\ \hline & +9x-27 \\ & 9x-27 \\ \hline & 0 \end{array}$$

Remark: remainder is 0

$$x^3-27=(x^2+3x+9)(x-3)+0$$

$$\Rightarrow \frac{x^3-27}{x-3}=x^2+3x+9+\frac{0}{x-3}$$

$$=\boxed{x^2+3x+9}.$$

8. Given $f(x)=\dfrac{x(x-2)(3x+1)}{(1+x)(2-x)\left(1-\frac{1}{x}\right)}$. Express $f(x)$ as a rational function then perform the long division to find the quotient and remainder.

Remark: Simplify first.

$$f(x)=\frac{x\cdot\cancel{(x-2)}^{-1}(3x+1)}{(1+x)\cancel{(2-x)}(1-\frac{1}{x})}=\frac{-x\cdot(3x+1)}{(1+x)(1-\frac{1}{x})}=\frac{-3x^2-x}{\cancel{1}+x-\frac{1}{x}\cancel{-1}}=\frac{(-3x^2-x)\cdot x}{(x-\frac{1}{x})\cdot x}=\frac{-3x^3-x^2}{x^2-1}.$$

$$\begin{array}{r|l} & -3x-1 \\ \hline x^2+0-1 & -3x^3-x^2+0+0 \\ & -3x^3+0+3x \\ \hline & -x^2-3x+0 \\ & -x^2+0+1 \\ \hline & -3x-1. \end{array}$$

$$(-3x^3-x^2)=(-3x-1)\cdot(x^2-1)-3x-1$$

$$\Rightarrow \frac{-3x^3-x^2}{x^2-1}=\boxed{-3x-1}+\frac{\boxed{-3x-1}}{x^2-1}.$$

Remember: The answers below are to help you check you work. The important thing is to be able to create and understand the complete solutions to these problems. Please re-read over the definitions/theorems/examples in the above notes as many times as necessary to gain a full understanding. Feel free to email your instructor or visit the MLC if you have questions. Typically on quizzes and exams the answer is worth very few points. The majority of the points are awarded on **the work** needed to get to the answer.

Answers

1. $341 + \frac{3}{4}$
2. $q(x) = 4x^2 + 9x + 19$, $r(x) = 39$
3. $q(x) = -2x - 2$, $r(x) = \cancel{-2}$ $+2$
4. $q(x) = x^2 - 2x$, $r(x) = 1$
5. $q(x) = x + 1$, $r(x) = 3x + 1$
6. $q(x) = 2$, $r(x) = \cancel{1}$ -1
7. $x^2 + 3x + 9, x \neq 3$
8. $q(x) = -3x - 1$, $r(x) = -3x - 1$